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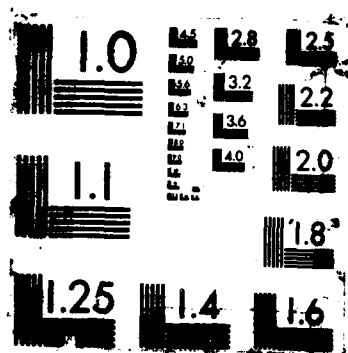
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The investigator was able to prove an optimal error estimate for approximating functions in sobolev spaces using a space of piecewise polynomial functions (based on the p-version of the finite element method). Optimal approximation results were also obtained for the h-p version of the finite element method using quasiuniform meshes. Papers accepted for publication during this period of effort included such titles as "The optimal convergence rate of the p-version of the finite element methods", "Some optimal approximation results with application to the h-p-, and h-p versions of the finite element method", and "The h-p version of the finite element method with quasi-uniform meshes".

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ANALYSIS OF THE PERFORMANCE OF MIXED FINITE ELEMENT METHODS

Principal Investigator: Manil Suri

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SUMMARY

The initial goal of this project is to analyze various mixed methods based on the p- and h-p versions of the finite element methods. The convergence of mixed methods depends on two factors: (1) Approximability of polynomial spaces used (2) Stability. In the past year, the question of approximability was thoroughly investigated and satisfactorily settled. In particular, optimal rates of convergence were proved for the approximation of both smooth and singular functions by the p-method, using continuous piecewise polynomials. These results were extended to the case of approximation by piecewise polynomials with k continuous derivatives. Analogous optimal convergence rates were established for the h-p version. The question of approximation of general inhomogeneous boundary conditions for elliptic systems of equations was also analyzed. In the area of stability, a mixed method based on boundary Lagrange multipliers for Laplace's equation was analyzed and a non-optimal convergence rate established.

In addition, finite element and finite difference schemes were analyzed for a non-linear steady-state reaction-diffusion equation and rates of convergence were established both theoretically and experimentally.



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Research Progress Report

The underlying goal of this project is the analysis of various theoretical and computational aspects of mixed finite element methods like stability, convergence, robustness and applicability. A significant portion of the research is geared towards the investigation of mixed methods based on the 'p' and 'h-p' versions of the finite element method. In particular, there are several mixed methods which have been thoroughly analyzed in the context of the h-version. Our goal is to investigate the convergence of such methods when the 'p' or 'h-p' extensions are used instead.

The convergence of any mixed method depends, in general, upon two basic factors:

- (I) Approximability of the finite-dimensional polynomial spaces being used.
- (II) Stability of the extension procedure employed (the "inf-sup" conditions).

Unlike the h-version, for which optimal approximability results are classical, only non-optimal results exist in the literature for the 'p' and 'h-p' versions. Hence, the first question considered by us was approximability. In this context, we have obtained the following results over the past year.

1. Optimal Approximation Results for the p-version using C^0 elements.

Previous results by Babuska, Szabo and Katz and by M. Dorr have shown that for a two-dimensional polygonal region Ω , the error e in approximating a function u ($u \in H^k(\Omega)$) using a space of piecewise polynomial functions of degree $\leq p$ converges to zero at the rate

$$(1) \quad \|e\|_{H^1} \leq C(\varepsilon) p^{-(k-1)+\varepsilon} \|u\|_{H^k}$$

where $\varepsilon > 0$ is arbitrarily small.

The proof of (1) indicates that the term $C(\varepsilon)$ can grow quickly with $\varepsilon \rightarrow 0$. Nevertheless, computational experience indicates that (1) holds without the

term τ , i.e., it suggests

$$(2) \quad \|e\|_{H^1} \leq C p^{-(k-1)} \|u\|_{H^k}.$$

A new method of analysis was used by us to prove the optimal result (2) in [1]. We also showed that when the solution has singular behaviour of the type $u \approx r^\alpha$, $\alpha > 0$ ((r, θ) being polar coordinates) and the vertex of the elements is at the origin, then

$$(3) \quad \|e\|_{H^1} \leq C p^{-2\alpha}$$

The above results also give an estimate for the rate of convergence obtained when the p-version is applied in practice to elliptic problems. In particular, (3) shows that the rate can be twice as much as that for the h-version with uniform mesh.

2. Extension to C^ℓ elements.

The results in [1] are not sufficient for problems like elasticity where C^1 elements must be used. Hence, in [5], these results were generalized to piecewise polynomial subspaces which are C^ℓ continuous ($\ell \geq 0$) over Ω . For such subspaces, the approximation rates corresponding to (2) and (3) become

$$(4) \quad \|e\|_{H^{\ell+1}} \leq C p^{-(k-\ell-1)} \|u\|_{H^k}$$

$$(5) \quad \|e\|_{H^{\ell+1}} \leq C p^{-2(\alpha-\ell)}$$

(4)-(5) improve upon the work by Katz and Wang and that by Dorr in which estimates optimal up to an arbitrary $\alpha > 0$ are proved. Moreover, these works use an interpolation assumption which is not assumed by us.

It should be remarked that our estimates (2) and (4) can easily be generalized to the case of three-dimensional domains as well.

3. Optimal Approximation Results for the h-p version using quasiuniform meshes.

The classical form of the error estimate for the h-version with quasiuniform mesh of size h is:

$$(6) \quad \|e\|_{H^1} \leq C(p)h^{n-1}\|u\|_{H^k}.$$

where p is the degree of polynomials used, $n = \min(k, p + 1)$ and the constant $C(p)$ depends upon p in an unspecified way. In [3], we showed that the actual error estimate obtained, which is optimal in both h and p , is of the form

$$(7) \quad \|e\|_{H^1} \leq Ch^{n-1}p^{-(k-1)}\|u\|_{H^k}$$

where n is in (6) and C is independent of h , p and u . Estimates were also proved for the case when the solution has singularities in the corners of the domain and various numerical examples illustrating these results were given.

4. The Treatment of General Inhomogeneous Boundary Conditions by the p-version.

Previous research by Babuska, Szabo and Katz for the application of the p -version to elliptic problems only treated the case when Dirichlet boundary conditions were either homogeneous or were polynomials of a certain degree. In [1], a method for approximating arbitrary Dirichlet boundary conditions based on an " H^1 projection" was presented. It was shown that this method yielded optimal convergence, provided the boundary function being approximated was in $H^{1+\epsilon}(\Gamma_i)$ for every segment Γ_i of the boundary for some $\epsilon > 0$. This method was incorporated into the p -version commercial code PROBE.

In [6], we showed that if the boundary function was only in $H^{k+\epsilon}(\Gamma_i)$, then the H^1 projection could lead to a non-optimal convergence rate. An alternate method, the " H^k projection," based on Tchebycheff expansions was presented and shown to be optimal. The results were extended to elliptic systems of equations with a general class of boundary conditions. In particular, it was shown that the elliptic system:

$$(8) \quad -\Delta u_1 + u_1 = f_1, \quad -\Delta u_2 + u_2 = f_2 \quad \text{in } \Omega$$

$$(9) \quad \alpha(x)u_1 + \beta(x)u_2 = g, \quad \alpha \geq 0, \beta \geq 0, \alpha + \beta > 0 \quad \text{on } \Gamma$$

$$(10) \quad -\beta(x)\frac{\partial u_1}{\partial n} + \alpha(x)\frac{\partial u_2}{\partial n} = h \quad \text{on } \Gamma$$

could be treated by the new method. Numerical results were provided.

The " $H^{\frac{1}{2}}$ projection" will be incorporated into the new version of PROBE to treat systems like (8)-(10) which arise in certain elasticity problems.

The approximability results mentioned above have been summarized in [2], together with related results obtained by other researchers. Essentially the questions that remained regarding the approximability of the p- and h-p versions have been answered, so that we have completed objective (I).

The next step is to analyze (II), the question of stability. In this area, so far we have studied a mixed method for Laplace's equation, based on the use of a Lagrange multiplier for the normal derivative on the boundary. We have shown in [7] that the stability constant grows at the rate of $p^{\frac{1}{2}}$ (leading to a loss in the expected rate of convergence by $p^{\frac{1}{2}}$). We are still trying to determine whether or not this is the best rate possible. During the next year, our efforts are expected to be concentrated on analyzing the stability of the p- and h-p versions of various mixed methods.

Finally, a project involving steady-state reaction-diffusion equations for the pth order isothermal reaction has been completed [4]. Our goal was to develop and analyze numerical schemes for the following problem, which has been studied by I. Stakgold and A. Friedman, among others:

$$(11) \quad \Delta u = \lambda u^p \text{ in } \Omega \quad u \geq 0 \text{ in } \Omega, \quad 0 < p < 1, \quad \lambda > 0$$

$$(12) \quad u = 1 \text{ on } \partial\Omega$$

The non-differentiability of the right hand side precludes the use of standard techniques of analysis. We have investigated both a finite element and finite difference approach to (11)-(12) and have proved an optimal rate of convergence for the case $p \geq \frac{1}{2}$. Further research in this area will involve analysis for the time-dependent version of (11)-(12).

Professional Personnel Associated with Research

The following graduate students from the Mathematics Department at UMBC were hired during the summer of 1986 to assist in computer programming:

T. Shen - worked on the approximation of boundary conditions for systems of equations by the p-method.

J. Liu - worked on two-dimensional finite element computations for reaction-diffusion equations.

In addition, the research on the p- and h-p versions has been done jointly with Dr. Ivo Babuska of University of Maryland College Park, while the research on reaction-diffusion problems has been done jointly with Dr. A. K. Aziz and Dr. B. Stephens of UMBC.

Papers Presented

- (1) Numerical Analysis Seminar, University of Maryland College Park, October, 1985, Optimal Convergence Results for the p-version.
- (2) Finite Element Circus at Bookhaven Labs, N.Y., November, 1985, Optimal convergence results for the h-p version.
- (3) "Methods of Functional Analysis in Approximation Theory" an international conference at I.I.T., Bombay, India, December, 1985, Optimal approximation results for the h-, p- and h-p versions.
- (4) Finite Element Circus at Rutgers University, New Jersey, April, 1986, Finite Element Method for reaction-diffusion equations.
- (5) ICM-86, Berkeley, California, August, 1986, Optimal error estimates for the h-, p- and h-p versions of the finite element method.

PUBLICATIONS

- [1] [with I. Babuska] "The Optimal Convergence Rate of the p-version of the Finite Element Methods," October, 1985 [To appear in SIAM Journal of Numerical Analysis, 1987].
- [2] "Some Optimal Approximation Results with Applications to the h-, p- and h-p versions of the Finite Element Method," December, 1985 [to appear in "Methods of Functional Analysis in Approximation Theory", Birkhauser-Verlag, 1986].
- [3] [with I. Babuska] "The h-p version of the Finite Element Method with Quasiuniform Meshes," April, 1986 [accepted for publication in RAIRO Mathematical Modelling and Numerical Analysis].
- [4] [with A. K. Aziz, B. Stephens] "Numerical Methods for Reaction-Diffusion Problems with Non-differentiable Kinetics" October, 1986 [to be submitted].
- [5] [with I. Babuska] "The p-version of the Finite Element Method for Elliptic Equations of Order $2k$ " [paper in preparation].
- [6] [with I. Babuska] "Treatment of General Boundary Conditions of Elliptic Systems by the p-version of the Finite Element Method [paper in preparation].
- [7] [with I. Babuska] "A Mixed p-version Finite Element Method for Elliptic Problems using Boundary Lagrange Multipliers" [research in progress].